The fundamental theorem of asset pricing under small transaction costs

Walter Schachermayer

University of Vienna
Faculty of Mathematics

Basic setting of Mathematical Finance:

$(S_t)_{0 \leq t \leq T}$ stochastic process modelling the price of a risky asset ("stock").

$B_t \equiv 1, \quad$ for $0 \leq t \leq T$: riskfree "bond".

Typical Question (Bachelier 1900, Black-Merton-Scholes 1973):

Pricing and Hedging of options like

$$C_T = (S_T - K)_+$$

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Basic Result:

Fundamental Theorem of Asset Pricing

Under suitable assumptions we have:
$(S_t)_{0 \le t \le T}$ does not allow for an arbitrage iff there is an equivalent martingale measure $Q \sim \mathbb{P}$ for S.

Ross '76
Harrison–Kreps '79
Harrison–Pliska '81
Kreps '81
.
.
.
Delbaen-S. '94,'98.

Corollary (sometimes called "second fundamental theorem of asset pricing"):

If there is a *unique equivalent martingale measure Q* for the process $(S_t)_{0 \le t \le T}$ then the option C_T above (in fact, any $\mathcal{F}_T$-measurable, Q-integrable function) can be represented as

$$C_T = \mathbb{E}_Q[C_T] + \int_0^T H_t \, dS_t,$$

for suitable "hedging strategy" $(H_t)_{0 \le t \le T}$.

Application:

$S_t = S_0 + \sigma W_t, \quad 0 \le t \le T$ (Bachelier 1900).
$S_t = S_0 e^{\sigma W_t + \mu t}, \quad 0 \le t \le T$ (Samuelson 1965).

Mathematical tool:

"Martingale representation theorem" (K. Itô).

Theorem

([Delbaen, S. 1994]): Let $(S_t)_{0\le t\le T}$ be a locally bounded process which fails to be a semi-martingale (e.g. fractional Brownian motion with $H \neq \frac{1}{2}$).
Then $(S_t)_{0\le t\le T}$ allows for a **free lunch with vanishing risk** *by simple integrands.*
More precisely: there is $\alpha > 0$ such that, for $\varepsilon > 0$ and $M > 0$, there is a simple integrand $H = \sum_{i=1}^{N} H_i \mathbb{1}_{]t_{i-1},t_i]}$ such that

$$(H\cdot S)_T \geq -\varepsilon, \qquad a.s$$

and

$$\mathbb{P}[(H\cdot S)_T \geq M] \geq \alpha.$$

Compare also Rogers '97, Cheridito '03, Sottinen-Valkeila '03.

<u>But:</u> If we introduce transaction costs of $\varepsilon > 0$, the arbitrage possibilities disappear in a wide class of models, containing (exponential) fractional Brownian motion.
[Guasoni, Rasonyi, Schachermayer '08]

Formal setting: Let $(S_t)_{0 \leq t \leq T}$ be an $\mathbb{R}_+$-valued stochastic process and $\varepsilon > 0$.

Assume that S is continuous.

ask price: $S_t(1+\varepsilon)$
bid price: $S_t/(1+\varepsilon)$

Davis-Norman ´90, Jouini-Kallal '95, Cvitanic-Karatzas '96, Kabanov, Stricker, Touzi, Rasonyi,....

Trading strategies:

Predictable processes $(\vartheta_t)_{0 \leq t \leq T}$ of *finite variation* and satisfying $\vartheta_0 = \vartheta_T = 0$: "trading strategy".

Value process:

$$V_t^{\varepsilon}(\vartheta) = \int_0^t \vartheta_u dS_u - \varepsilon \int_0^t S_u \, d\mathrm{Var}_u(\vartheta)$$

well defined a.s. as a pathwise Stieltjes integral.

Campi, S. 2006 show that this forms indeed the *natural* class of integrands.

Admissibility of value processes:

Two versions of admissibility:

Version A (Harrison-Pliska ´81,...Delbaen-S. ´94,´98)

$$V_t^{\varepsilon}(\vartheta) \geq -M \quad \text{a.s.},$$

for each $0 \leq t \leq T$ and some $M > 0$.

Version B (Merton ´73,...,Sin ´96, Yan ´98, Jarrow-Protter-Shimbo ´08)

$$V_t^{\varepsilon}(\vartheta) \geq -M(1 + S_t) \quad \text{a.s.},$$

for each $0 \leq t \leq T$ and some $M > 0$.

Definition

The stochastic process $(S_t)_{0 \leq t \leq T}$ allows for an arbitrage under ε transaction costs (for $\varepsilon > 0$ fixed) if there is an admissible value process $(V_t^{\varepsilon}(\vartheta))_{0 \leq t \leq T}$ s.t.

$$\mathbb{P}[V_T^{\varepsilon}(\vartheta) \geq 0] = 1,$$

$$\mathbb{P}[V_T^{\varepsilon}(\vartheta) > 0] > 0.$$

Remark

Depending on the choice of the concept of admissibility there are presently two versions of the concept of (no) arbitrage.

The analogue to the concept of equivalent (local) martingale measures:

Definition (Jouini-Kallal '95,...)

An ε-consistent price system for the given process $(S_t)_{0\le t\le T}$ is a pair $((\tilde{S}_t)_{0\le t\le T}, Q)$ s.t. $\tilde{S}$ is an $\mathbb{R}_+$-valued stochastic process satisfying

(i) $\frac{1}{1+\varepsilon} \le \frac{\tilde{S}_t}{S_t} \le 1+\varepsilon$, a.s. for all $0 \le t \le T$,

(ii) $Q \sim \mathbb{P}$,

(iii) Version A: $(\tilde{S}_t)_{0\le t\le T}$ is a local martingale under Q.
Version B: $(\tilde{S}_t)_{0\le t\le T}$ is a true martingale under Q.

Theorem

(Guasoni-Rasonyi-S. 2008): Let $(S_t)_{0 \leq t \leq T}$ *be an* $\mathbb{R}_+$*-valued continuous stochastic process adapted to* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$*. T.F.A.E.*

(i) For each $\varepsilon > 0$*,* S *does not allow for an arbitrage under* ε *transaction costs.*

(ii) For each $\varepsilon > 0$*,* S *admits an* ε*-consistent price system.*

Remark

Remark: The theorem holds true in Version A as well as in Version B.

Proof of Theorem: (sketch of ideas)

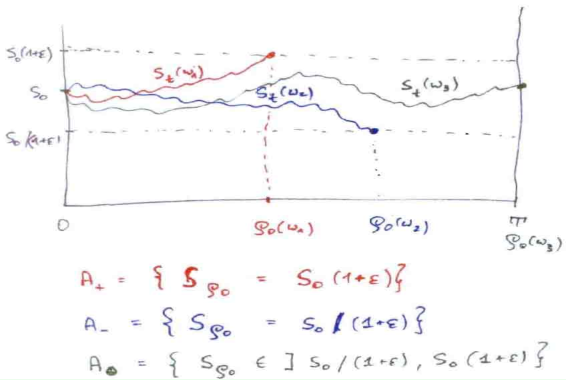
(ii) $\Rightarrow$ (i) easy (as usual):
Make the easy observation that it is better to trade on $(\tilde{S}_t)_{0 \leq t \leq T}$, without transaction costs, than to trade on $(S_t)_{0 \leq t \leq T}$ with ε transaction costs because of

$$S_t/(1+\varepsilon) \leq \tilde{S}_t \leq S_t(1+\varepsilon).$$

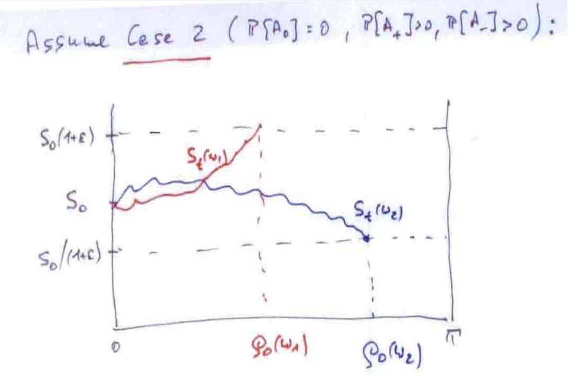
(i) $\Rightarrow$ (ii) is the non-trivial part of the theorem. Assuming *NA* under ε transaction costs, let us construct $\tilde{S}$ and Q.

Define the stopping time ρ_0 by

$$\rho_0 = \inf\{t : \ \frac{S_t}{S_0} \text{ equals } 1+\varepsilon \text{ or } \frac{1}{1+\varepsilon}\} \wedge T$$



The subsequent analysis reduces to the following cases:
Case 1: $\mathbb{P}[A_+] > 0, \quad \mathbb{P}[A_-] > 0, \quad \mathbb{P}[A_0] > 0.$
Case 2: $\mathbb{P}[A_+] > 0, \quad \mathbb{P}[A_-] > 0, \quad \mathbb{P}[A_0] = 0.$



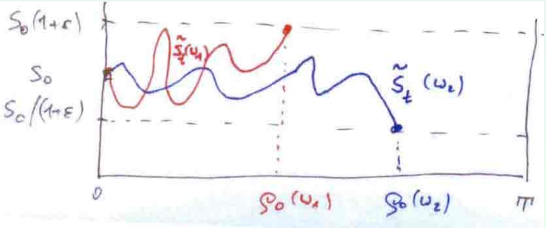
Define the desired measure $Q \sim \mathbb{P}$ on $\mathcal{F}_{\rho_0}$ in such a way that $Q[A_+] = \frac{1}{2+\varepsilon}$ and $Q[A_-] = \frac{1+\varepsilon}{2+\varepsilon}$.
Define $(\tilde{S}_t)_{0 \leq t \leq \rho_0}$ by letting

$$\tilde{S}_t = \mathbb{E}_Q[S_{\rho_0} | \mathcal{F}_t], \quad 0 \leq t \leq \rho_0.$$

and observe that

$$\tilde{S}_0 = Q[A_+] S_0 (1+\varepsilon) + Q[A_-] S_0/(1+\varepsilon) = S_0$$

and that $(\tilde{S}_t)_{0 \leq t \leq \rho_0}$ remains in the "ε-corridor"



The inequality $\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_t}{S_t} \leq 1+\varepsilon$ then is satisfied for $0 \leq t \leq \rho_0$, and $(\tilde{S}_t)_{0 \leq t \leq \rho_0}$ is a Q-martingale.

Idea of continuation of construction:
As $\tilde{S}_{\rho_0} = S_{\rho_0}$ we may iterate the procedure by letting

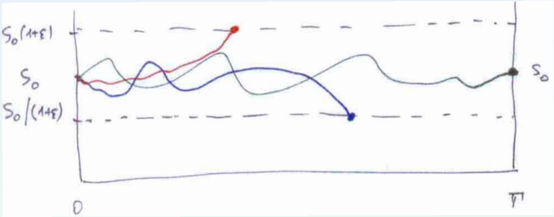
$$\rho_1 = \inf\{t \geq \rho_0 : \quad \frac{S_t}{S_{\rho_0}} \text{ is either } 1+\varepsilon \ \text{ or } \ \frac{1}{1+\varepsilon}\} \wedge T$$

etc, etc.

Let us now turn to

Case 1: $(\mathbb{P}[A_0] > 0, \mathbb{P}[A_+] > 0, \mathbb{P}[A_-] > 0)$.

Assume (essentially w.l.g.) that $S_T = S_0$ on A_0. We now have one degree of freedom in the construction of Q.



To define Q, choose $0 < \lambda < 1$, and let

$$Q[A_0] = \lambda, \quad Q[A_+] = (1-\lambda)\frac{1}{2+\varepsilon}, \quad Q[A_-] = (1-\lambda)\frac{1+\varepsilon}{2+\varepsilon}.$$

$$\Rightarrow \tilde{S}_0 = \mathbb{E}_Q[S_{\rho_0}] = S_0$$

Remark

If S has "conditional full support" in $C([0,T],\mathbb{R}_+)$ w.r.t. to $\|\cdot\|_\infty$, then we are always in case 1 of the above construction and therefore have in every step one (conditional) degree of freedom $0<\lambda<1$.

This allows for the construction of "many" ε-consistent price systems $(\tilde{S}, Q)$. These may e.g. be used to give easy "dual proofs" of the so-called "face lifting" theorems (Soner, Shreve, Cvitanic '95, Levental, Skorohod '97).

Face Lifting Theorem (Levental-Skorohod '96, Soner-Shreve-Cvitanic '95,...,Guasoni-Rasonyi-S. '08):

Suppose that $S = (S_t)_{0 \le t \le T}$ has conditional full support in $C_+[0, T]$ and suppose $\varepsilon > 0$ as transaction costs.
Then the cheapest way to superreplicate an option $C_T = (S_T - K)_+$, i.e., the smallest constant such that there is H satisfying

$$C_T \le \text{ constant } + \int_0^T H_t \, dS_t - \varepsilon \int_0^T S_t \, d\mathrm{Var}_t(\vartheta)$$

is to take

$$\text{constant } = S_0, \qquad H_t \equiv 1.$$

Summing up:

In the presence of (even very small) transaction costs, the paradigm of replication/super-replication cannot provide any non-trivial information for the problem of pricing and hedging derivatives.

What to do?

- Utility maximisation (portfolio optimisation) does make good sense also in the presence of transaction costs:

$$u(x) = \sup_{\vartheta} \mathbb{E}[U(x + \int_0^T \vartheta_t \, dS_t - \varepsilon \int_0^T S_t \, d\mathrm{Var}_t(\vartheta))], x \in \mathbb{R}_+.$$

  where $U(x)$ is a fixed concave, increasing function (e.g. $U(x) = \log(x)$.)
- This problem still makes sense for "random endowment" $X_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ (e.g. $X_T = C_T$):

$$u(X_T) = \sup_{\vartheta} \mathbb{E}[U(X_T + \int_0^T \vartheta_t \, dS_t - \varepsilon \int_0^T S_t \, d\mathrm{Var}_t(\vartheta))]$$

- Utility indifference pricing (de Finetti: "certainty equivalent"): define the price x for X_T implicitly by

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What to do?

- Let $\hat{\vartheta}^{x}$ and $\hat{\vartheta}^{X_T}$ be the optimizing strategies corresponding to x and X_T; the difference $\hat{\vartheta}^{X_T} - \hat{\vartheta}^{x}$ may be interpreted as a hedging strategy for X_T.
- Research programm:
  derive an asymptotic expansion for $\varepsilon \to 0$ and $H \to \frac{1}{2}$ how the option prices and hedging strategies deviate from the classical Black-Scholes price (compare Fouque-Papanicolao-Sircar, Janecek-Shreve, Kramkov-Sirbu etc.).

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